SHENTON COLLEGE

Examination Semester One 2019
Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section One
(Calculator-free)

Your name SOLUTIONS

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section One.
Formula sheet.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters
Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Student Score
Section One: Calculator-free	8	8	50	52	
Section Two: Calculator-assumed	13	13	100	98	
			Total	150	

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Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: if you need to use the space to continue an answer, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than 2 marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

It is recommended that you **do not use pencil, except in diagrams**.

STRUCTURE OF THIS PAPER

QUESTION	MARKS AVAILABLE	MARKS AWARDED
1	6	
2	4	
3	6	
4	6	
5	7	
6	7	
7	8	
8	8	
TOTAL	**52**	

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine the modulus and argument of $\frac{3}{1-i}$. (3 marks)

$\frac{3}{1-i} \times \frac{1+i}{1+i} = \frac{3+3i}{2}$

✓ z in a + bi form

$z = \frac{3}{2} + \frac{3}{2}i$

✓ Arg

$\Rightarrow \text{Arg}(z) = \frac{\pi}{4}$

✓ |z|

$|z| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$

$= \sqrt{\frac{18}{4}}$

$= \frac{3\sqrt{2}}{2}$

(b) Determine z^2 in the form $a + bi$, where $a, b \in \mathbb{R}$, when $z = 4\cos\left(\frac{\pi}{6}\right) + 4i\sin\left(\frac{\pi}{6}\right)$. (3 marks)

$z = 4 \text{ cis } \frac{\pi}{6}$

$z^2 = 16 \text{ cis } \frac{\pi}{3}$

Use of DeMoivre's on

$= 16\cos\frac{\pi}{3} + 16i\sin\frac{\pi}{3}$

✓ modulus

$= 16\left(\frac{1}{2}\right) + 16i\left(\frac{\sqrt{3}}{2}\right)$

✓ arguement.

$= 8 + 8\sqrt{3}i$

✓ a + bi form

Question 2 **(4 marks)**

The equations of three planes are shown below.

$$x - y + 3z = 11$$
$$x + 2y - 2z = 0$$
$$x - y + z = 9$$

(a) Determine the coordinates of the point of intersection of the planes. (3 marks)

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 11 \\ 1 & 2 & -2 & 0 \\ 1 & -1 & 1 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 11 \\ 0 & 3 & -5 & -11 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$z = 1$

$y = -2$

$x = 6$

Intersection at $(6, -2, 1)$

✓ method to solve for 1st variable

✓ substitute / solve for complete set of variables

✓ answer as coordinate

(b) Determine the distance of the point of intersection of the planes from the origin. (1 mark)

$d = \sqrt{6^2 + (-2)^2 + 1^2}$

$= \sqrt{41}$

Question 3 **(6 marks)**

(a) State whether the planes with equations $2x - y + z = 2$ and $x + 3y + 2z = 1$ are perpendicular. Justify your answer. (2 marks)

Normal vectors are $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\\2\end{pmatrix}$ respectively.

$$\begin{pmatrix}2\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\3\\2\end{pmatrix} = 2 - 3 + 2$$
$$= 1.$$

✓ use dot product on normals

Normals not $\perp$ $(\tilde{a} \cdot \tilde{b} \neq 0)$,

$\therefore$ planes not $\perp$.

✓ interpret result.

(b) Determine the Cartesian equation of the plane that passes through the three points with position vectors shown below. (4 marks)

$$\mathbf{a} = \begin{pmatrix}1\\2\\0\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}3\\0\\1\end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix}2\\2\\2\end{pmatrix}$$

$\tilde{b} - \tilde{a} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$ $\quad \tilde{c} - \tilde{b} = \begin{pmatrix}-1\\2\\1\end{pmatrix}$ $\quad$ [or $\begin{pmatrix}1\\0\\2\end{pmatrix}$ or their negatives]

✓ any two vectors in the plane

$$\begin{matrix} i & j & k & i & j \\ 2 & -2 & 1 & 2 & -2 \\ -1 & 2 & 1 & -1 & 2 \end{matrix}$$

✓ cross product.

$(-2-2)i + (-1-2)j + (4-2)k.$

$$\begin{pmatrix}2\\-2\\1\end{pmatrix} \times \begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}-4\\-3\\2\end{pmatrix}.$$

✓ dot product with any point

$$\begin{pmatrix}-4\\-3\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} = -4 - 6$$
$$= -10$$

✓ Cartesian equation.

Plane $-4x - 3y + 2z = -10$

or $4x + 3y - 2z = 10$

Question 4 **(6 marks)**

Functions f and g are defined over their natural domains by $f(x) = \sqrt{8 - x}$ and $g(x) = 3 + \frac{4}{\sqrt{x}}$.

(a) State the domain of

(i) $g(x)$. (1 mark)

$x > 0$

(ii) $g^{-1}(x)$. (2 marks)

Domain $g^{-1}(x)$ = Range $g(x)$ ✓ link domain and range

$x > 3$ ✓ answer

(b) Determine $f \circ g(x)$ and the natural domain of this composite function. (3 marks)

$$f(g(x)) = \sqrt{8 - \left(3 + \frac{4}{\sqrt{x}}\right)}$$

$$= \sqrt{5 - \frac{4}{\sqrt{x}}}$$ ✓ composition.

$x \geq 0$ and $\frac{4}{\sqrt{x}} \leq 5$ ✓ conditions for non-negative $\sqrt{\ }$

$$\frac{4}{5} \leq \sqrt{x}$$

$$\frac{16}{25} \leq x$$ ✓ Domain.

Domain $f(g(x)) = \left\{x : x \in \mathbb{R}, \underbrace{x \geq \frac{16}{25}}_{\text{accept just this.}}\right\}$

Question 5 **(7 marks)**

Four functions are defined as

$$f(x) = x^2 + 4x - 5, \qquad g(x) = 3x^2 + 2x - 1, \qquad h(x) = x + 5, \qquad k(x) = x - 1$$

$(x+5)(x-1)$ $(x+1)(3x-1)$

Determine the equations of all asymptotes of the following graphs.

(a) $y = \dfrac{h(x)}{f(x)}$. (2 marks)

$= \dfrac{x+5}{(x+5)(x-1)}$

$= \dfrac{1}{x-1}, \quad x \neq -5$

vertical asymptote $x = 1$
horizontal asymptote $y = 0$

✓ $x = 1$
✓ $y = 0$

(b) $y = \dfrac{f(x)}{g(x)}$. $= \dfrac{x^2+4x-5}{3x^2+2x-1}$ (2 marks)

$= \dfrac{(x+5)(x-1)}{(x+1)(3x-1)}$

As $x \to \pm\infty$, $y = \dfrac{1}{3}$

vertical asymptotes at $x = -1, x = \frac{1}{3}$
horizontal asymptote $y = \frac{1}{3}$

✓ $x = -1, \frac{1}{3}$
✓ $y = \frac{1}{3}$

(c) $y = \dfrac{g(x)}{k(x)}$. (3 marks)

$= \dfrac{3x^2+2x-1}{x-1}$

3	2	-1	1
	3	5	
3	5	4	

$= 3x + 5 + \dfrac{4}{x-1}$

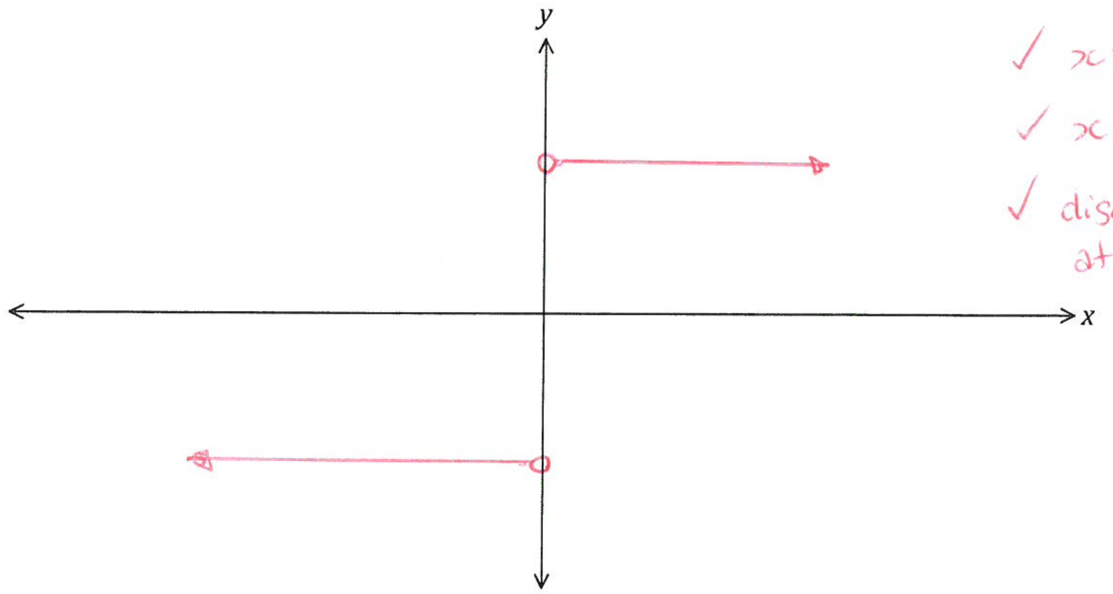
vertical asymptote $x = 1$
oblique asymptote $y = 3x + 5$

✓ poly + remainder form
✓ $x = 1$
✓ $y = 3x + 5$

Question 6 (7 marks)

(a) On the axes below, sketch the graph of $y = \dfrac{2x}{|x|}$. (3 marks)

$y = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}$

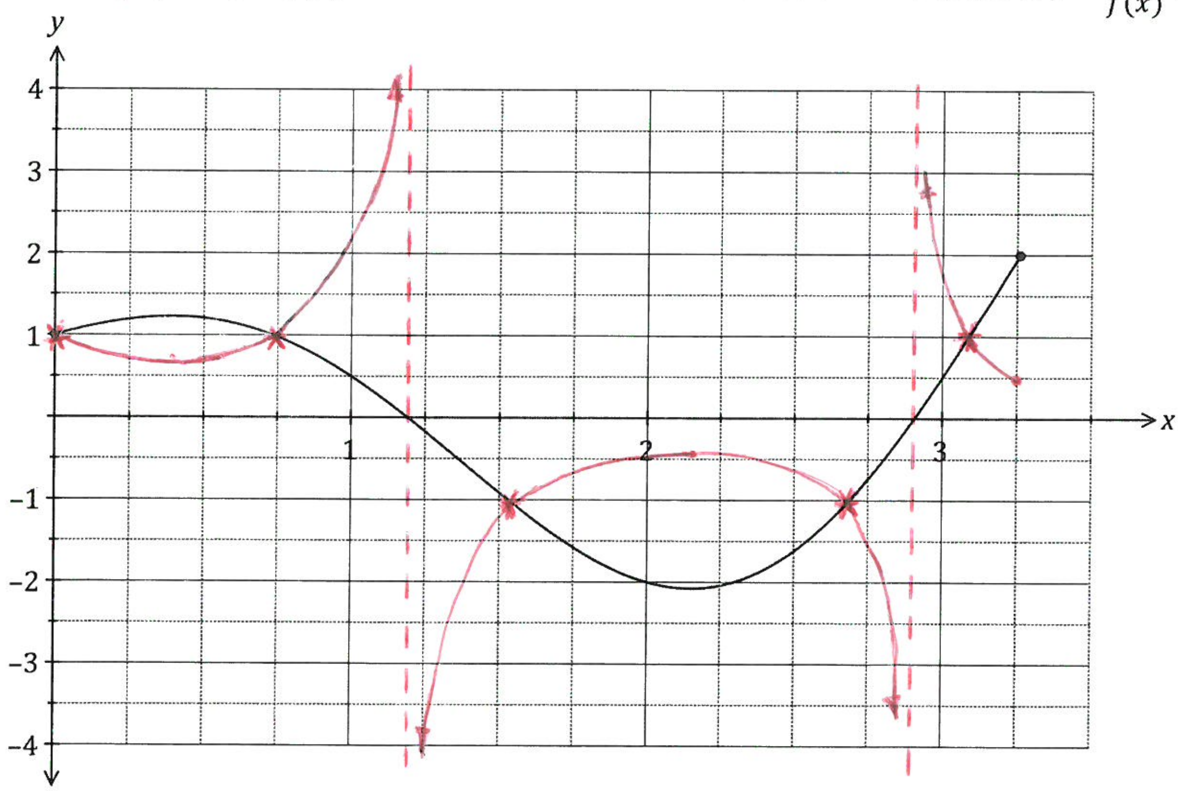


✓ $x > 0$

✓ $x < 0$

✓ discontinuities at $x = 0$

(b) The graph of $y = f(x)$ is shown below. On the same axes draw the graph of $y = \dfrac{1}{f(x)}$. (4 m



✓ max & min align with those of $f(x)$

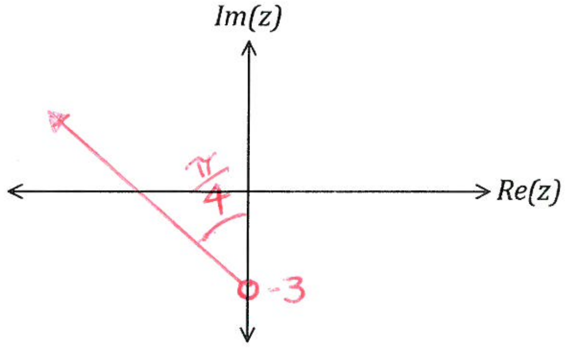
✓ behavior at asymptotes & end of domain

✓ vertical asymptotes

✓ intersection with $f(x)$ at $y = \pm 1$.

Question 7 **(8 marks)**

(a) Sketch the locus of points z in the complex plane determined by $\arg(z+3i) = \frac{3\pi}{4}$. (3 marks)



✓ ray from $-3i$

✓ open circle at start of ray.

✓ indicates angle / both scales

(b) Another locus of points z in the complex plane is determined by $z\bar{z} + z + \bar{z} = 8$.

(i) Show that this locus can also be defined in the form $|z - w| = k$, clearly showing the value of constant w and the value of constant k. (3 marks)

Let $z = x + iy$

$(x+iy)(x-iy) + x+iy + x-iy = 8$

$x^2 + y^2 + 2x = 8$

$(x+1)^2 - 1 + y^2 = 8$

$(x+1)^2 + y^2 = 9.$

Circle with centre $(-1, 0)$ and radius 3

Hence $|z - (-1)| = 3$

$w = -1, \ k = 3$

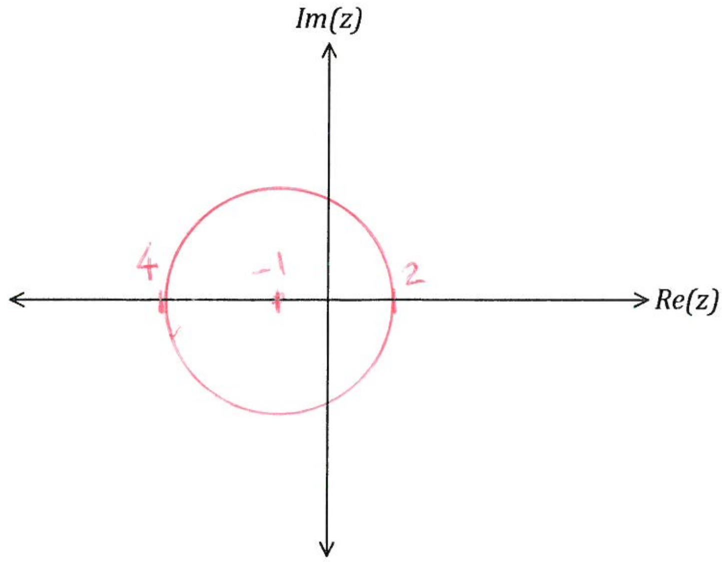
✓ establish eqⁿ of circle in Cartesian form

✓ complete square to identify centre & radius

✓ write in magnitude form

$|z+1| = 3$ is ok.

(ii) Sketch the locus on the axes below. (2 marks)



✓ circle centered at -1

✓ indicates radius (scale or otherwise)

Question 8 **(8 marks)**

Let $z = x + yi$ and $z^2 = a + bi$ where $a, b, x, y \in \mathbb{R}$.

(a) Show that $\sqrt{a^2 + b^2} + a = 2x^2$. (4 marks)

if $z = x + iy$, then $z^2 = x^2 + 2xyi - y^2$ ✓ z^2

and $z^2 = a + bi \Rightarrow a = x^2 - y^2,\ b = 2xy$

✓ equate real and imaginary for a & b in terms of x & y

$\sqrt{a^2+b^2} + a$

$= \sqrt{(x^2-y^2)^2 + (2xy)^2} + x^2 - y^2$

$= \sqrt{x^4 - 2x^2y^2 + y^4 + 4x^2y^2} + x^2 - y^2$

✓ substitute & expand a^2 & b^2

$= \sqrt{x^4 + 2x^2y^2 + y^4} + x^2 - y^2$

$= \sqrt{(x^2+y^2)^2} + x^2 - y^2$

$= x^2 + y^2 + x^2 - y^2$

$= 2x^2$

✓ factorise, simplify & show

(b) By solving the equation $z^4 - 16z^2 + 100 = 0$ for z^2 or otherwise, determine the roots of the equation in Cartesian form. (4 marks)

Let $z^2 = w$

$w^2 - 16w + 100 = 0$

By Q.F.

$\dfrac{16 \pm \sqrt{256 - 400}}{2}$

$= \dfrac{16 \pm \sqrt{-144}}{2}$

$= 8 \pm 6i$

$w = 8 \pm 6i$

ie $z^2 = 8 + 6i$

or $z^2 = 8 - 6i$

✓ solve for z^2

✓ use $\sqrt{a^2+b^2} + a$ to solve for x

✓ use $a = x^2 - y^2$ to solve for y

✓ list four solutions for z.

$z^2 = 8 + 6i$

$a = 8,\ b = 6$

$\sqrt{64 + 36} + 8$

$= 18$

$2x^2 = 18$

$x = \pm 3$

$z^2 = 8 - 6i$

$a = 8,\ b = -6$

$\sqrt{64 + 36} + 8$

(probably didn't need this part then...)

$a = x^2 - y^2$

$8 = 9 - y^2$

$y = \pm 1$

$\boxed{z = 3+i,\ 3-i,\ -3+i,\ -3-i}$

End of questions

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Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two
(Calculator-assumed)

Your name SOLUTIONS

Time allowed for this section

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Working time for paper: 100 minutes

Material required/recommended for this section

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Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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STRUCTURE OF THIS PAPER

QUESTION	MARKS AVAILABLE	MARKS AWARDED
9	7	
10	10	
11	6	
12	6	
13	9	
14	7	
15	8	
16	7	
17	8	
18	8	
19	8	
20	8	
21	6	
TOTAL	**98**	

Section Two: Calculator-assumed (98 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 (7 marks)

(a) Determine the values of the real constant a and the real constant b given that $z - 4 + 2i$ is a factor of $z^3 + az + b$.

(4 marks)

When $z = 4-2i$,

$z^3 + az + b = 0$ ✓ identify root.

$(4-2i)^3 + a(4-2i) + b = 0$ ✓ substitute

$16 - 88i + 4a - 2ai + b = 0$ ✓ create equations from real & imaginary

$\rightarrow$ $16 + 4a + b = 0$

$-88 - 2a = 0$ ✓ a & b

$a = -44, \ b = 160$

(b) Clearly show that $2 + i$ is a root of the equation $z^3 - 7z^2 + 17z - 15 = 0$. (2 marks)

$(2+i)^3 - 7(2+i)^2 + 17(2+i) - 15$ ✓ indicate substitution

$= (8 + 12i - 6 - i) - 7(4 + 4i - 1) + 17(2+i) - 15$

$= (2 + 11i) - (21 + 28i) + (34 + 17i) - 15$ ✓ show some expanded forms

$= 2 - 21 + 34 - 15 + (11 - 28 + 17)i$

$= 0$

(c) State all three solutions of $z^3 - 7z^2 + 17z - 15 = 0$. (1 mark)

$z = 2+i, \ 2-i, \ 3$

Question 10 **(10 marks)**

(a) Consider the system of simultaneous equations… (6 marks)

$$2x - 4y + 2z = 8$$
$$-x + 5y + z = -9$$
$$x + y + (p^2 + 2)z = p$$

(i) Express the system as an augmented matrix and use row reduction techniques so that the coefficients of x and y in the third equation above are both zero

$$\left[\begin{array}{ccc|c} 2 & -4 & 2 & 8 \\ -1 & 5 & 1 & -9 \\ 1 & 1 & p^2+2 & p \end{array}\right]$$

✓ augmented matrix

✓ reduction on C_1

✓ reduction on C_2

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 3 & 2 & -5 \\ 0 & 3 & p^2+1 & p-4 \end{array}\right] \begin{array}{l} R_1' = \frac{1}{2}R_1 \\ R_2' = R_2 + R_1' \\ R_3' = R_3 - R_1' \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 3 & 2 & -5 \\ 0 & 0 & p^2-1 & p+1 \end{array}\right] R_3'' = R_3' - R_2'$$

Hence determine value/s for p so that the system has:

(ii) No solutions

$p^2 - 1 = 0$ but $p + 1 \neq 0$

$p = 1$

(iii) An infinite number of solutions

$p^2 - 1 = 0$ and $p + 1 = 0$

$p = -1$

(iv) A unique solution

$p \in \mathbb{R}, p \neq 1, p \neq -1$

(b) Determine whether the following system of equations has a unique solution, an infinite number of solutions, or no solutions. Provide a brief geometric interpretation of your findings. (3 marks)

$$x + 2y + z = 3$$
$$2x + 4y + 2z = 7$$
$$2x + y + 2z = 4$$

Equations 1 & 2 represent parallel planes (normals $\begin{pmatrix}1\\2\\1\end{pmatrix}$ & $\begin{pmatrix}2\\4\\2\end{pmatrix}$). Therefore the system has no solutions.
Since Equation 3 is not parallel to the others, this represents two parallel planes cut by a 3rd plane.

✓ indicates no solutions

✓ some justification (reduction or otherwise)

✓ geometric interpretation

(c) Describe the geometric interpretation of a system that has an infinite number of solutions. (1 marks)

Any of...

3 planes that all intersect along a common line.

2 coincident planes and one non-parallel plane

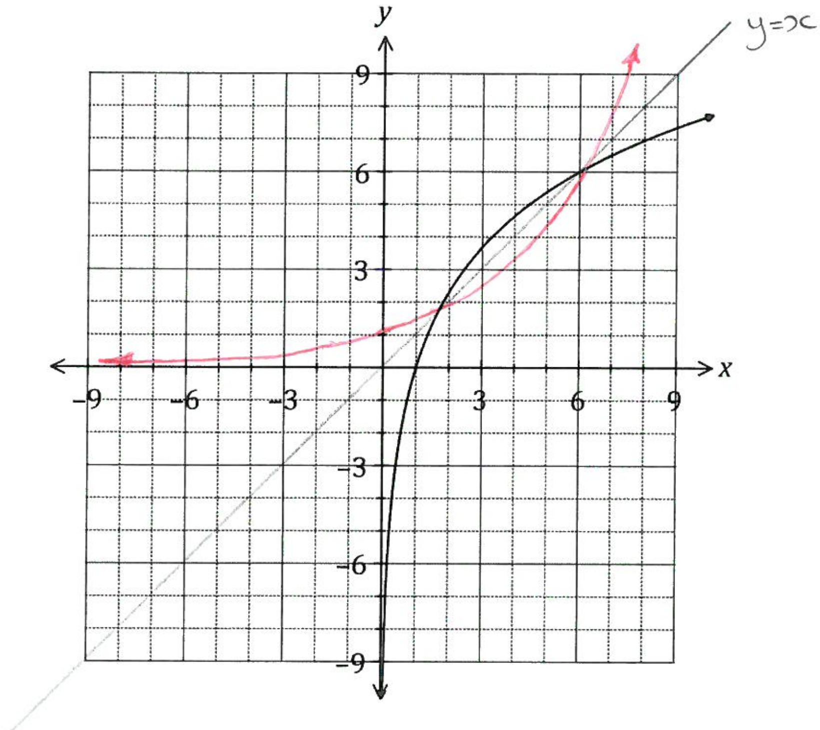
3 coincident planes

Question 11 **(6 marks)**

(a) Explain why the function $f(x) = \cos x$, where $x \in \mathbb{R}$, is not one-to-one. (1 mark)

Because there are many x values which generate the same y value, eg $\cos(0) = 1$, $\cos(2\pi) = 1$, etc.

(b) The graph of $y = g(x)$ is shown below. Sketch the graph of $y = g^{-1}(x)$ on the same axes. (2 marks)



✓ reflection about $y = x$

✓ intersections and axes

(c) The inverse function of h is defined as $h^{-1}(x) = x^2 - 8x + 17$ for $x \leq 4$. Determine the defining rule for $h(x)$ and state its domain. (3 marks)

$x = (y-4)^2 + 1$

$y = 4 \pm \sqrt{x-1}$

Domain $h^{-1}(x)$ = range $h(x)$

$\Rightarrow y \leq 4$

$\Rightarrow h(x) = 4 - \sqrt{x-1}$

Domain $h(x)$ $x \geq 1$

✓ inverse method (or otherwise) for $h(x)$

✓ choose appropriate $\pm$ based on statement of range

✓ domain of $h(x)$

Question 12 **(6 marks)**

The line **L** passes through the points **a** and **b,** given by the position vectors $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ respectively. A third point **p** is located at $\mathbf{i} + \mathbf{j} + \mathbf{k}$. Determine the position of the point on **L** that is closest to **p** and calculate this minimum distance. (6 marks)

$$\overrightarrow{ab} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

✓ equation for L

$$L = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

✓ point on L relative to p

Let $\tilde{r}$ be the point on L closest to $\tilde{p}$,

then $\overrightarrow{pr} \cdot \overrightarrow{ab} = 0$

✓ dot product = 0

$$\overrightarrow{pr} = \begin{pmatrix} 1 + \lambda \\ -2 + 2\lambda \\ 1 - 3\lambda \end{pmatrix}$$

✓ solve for λ

$$\begin{pmatrix} 1 + \lambda \\ -2 + 2\lambda \\ 1 - 3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 0$$

✓ position on L

✓ distance

$$1 + \lambda - 4 + 4\lambda - 3 + 9\lambda = 0$$

$$14\lambda - 6 = 0$$

$$\lambda = \frac{3}{7}$$

$$\tilde{r} = \frac{1}{7}\begin{pmatrix} 17 \\ -1 \\ 5 \end{pmatrix}$$ closest point

$$|\overrightarrow{pr}| = \frac{2\sqrt{42}}{7}$$ closest distance

Question 13 **(9 marks)**

Let $w = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(a) Express w, w^2, w^3 and w^4 in the form $r \operatorname{cis} \theta, -\pi < \theta \leq \pi$. (2 marks)

$w = \operatorname{cis}\left(\frac{-2\pi}{3}\right)$

$w^2 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$

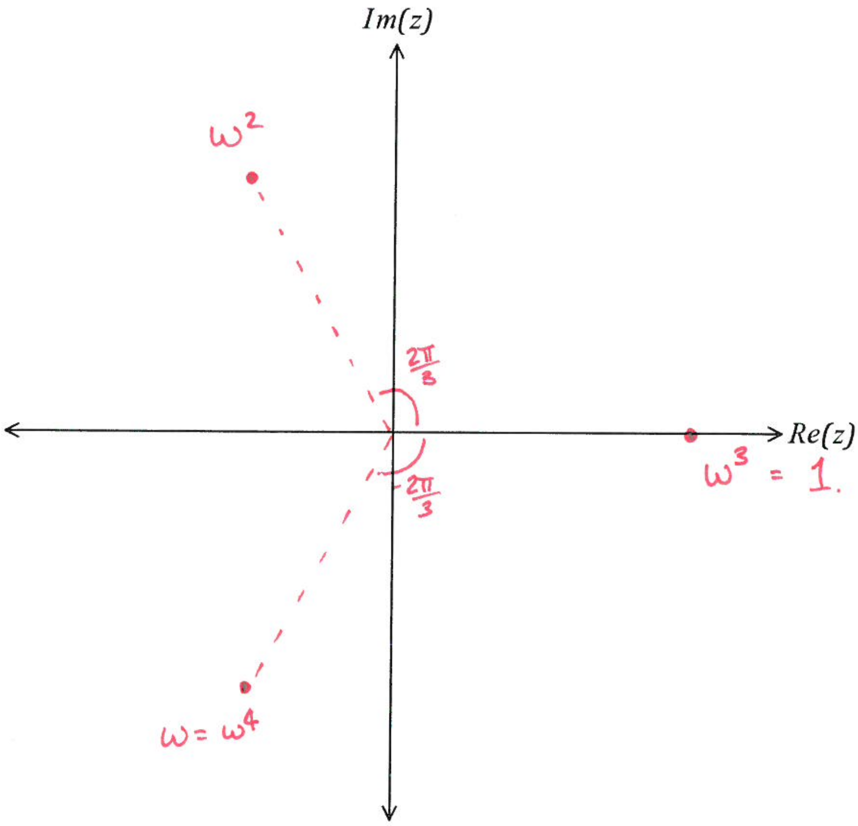
$w^3 = \operatorname{cis}(0)$

$w^4 = \operatorname{cis}\left(\frac{-2\pi}{3}\right)$

✓ w

✓ complete set.

(b) Sketch w, w^2, w^3 and w^4 on the Argand diagram below. (2 marks)



✓ points with rotational symmetry

✓ w^3 on positive real axis

(need some indication of scale for 2 marks)

(c) Describe the transformation in the complex plane of any point z when it is multiplied by w. (2 marks)

Rotation about the origin (clockwise) by $\frac{2\pi}{3}$

✓ rotation.

✓ include direction & magnitude.

(d) Simplify

(i) $w^0 + w^1 + w^2$. (1 mark)

0

(ii) $w^0 + w^1 + w^2 + \cdots + w^{2018} + w^{2019}$. (2 marks)

$2019 \div 3 = 673$

$$\left.\begin{array}{l} \omega^0 \\ + \omega^1 + \omega^2 + \omega^3 \\ + \omega^4 + \omega^5 + \omega^6 \\ \vdots \\ + \omega^{2017} + \omega^{2018} + \omega^{2019}. \end{array}\right]$$

673 triplets with a sum of zero

$= \omega^0$

$= 1.$

Question 14 **(7 marks)**

(a) Solve the equation $z^5 + 32 = 0$, writing your solutions in polar form $r \operatorname{cis} \theta$. (4 marks)

$z^5 = -32$

$= 2^5 \operatorname{cis} \pi$

✓ polar form for z^5

$z_1 = 2 \operatorname{cis} \frac{\pi}{5}$ rotational symmetry $\frac{2\pi}{5}$

$z_2 = 2 \operatorname{cis} \frac{3\pi}{5}$

$z_3 = 2 \operatorname{cis} \pi$

$z_4 = 2 \operatorname{cis} \frac{-\pi}{5}$

$z_5 = 2 \operatorname{cis} \frac{-3\pi}{5}$

✓ Use of DeMoivre's for one solution

✓ 5 solutions

✓ correct domain

(b) Use your answers from (a) to show that $\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$. (3 marks)

Since $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$\Rightarrow Re(z_1) + Re(z_2) + Re(z_3) + Re(z_4) + Re(z_5) = 0$

$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} + 2\cos\pi + 2\cos\frac{-\pi}{5} + 2\cos-\frac{3\pi}{5} = 0$

✓ sum of Reals = 0

$\cos(-\theta) = \cos(\theta)$ and $\cos\pi = -1$

✓ use of $\cos(-\theta)$ and $\cos\pi = -1$

$\therefore 2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 2 + 2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 0$

$4\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) = 2$

$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

✓ simplify & show

Question 15 **(8 marks)**

The position vectors of two particles at time t are given below, where a is a constant.

$$\mathbf{r}_A = 8\mathbf{i} - 5\mathbf{j} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \text{ and } \mathbf{r}_B = 3\mathbf{i} + a\mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

The paths of the particles cross at P but the particles do not meet.

(a) Determine the value of the constant a and the position vector of P. (5 marks)

$$\tilde{r}_A = \begin{pmatrix} 8+\lambda \\ -5+2\lambda \\ -1-\lambda \end{pmatrix} \qquad \tilde{r}_B = \begin{pmatrix} 3+3\mu \\ a-\mu \\ 1-2\mu \end{pmatrix}$$

✓ parameters to replace t

For paths to cross

$8+\lambda = 3+3\mu \quad \cdots ①$

$-5+2\lambda = a-\mu \quad \cdots ②$

$-1-\lambda = 1-2\mu \quad \cdots ③$

✓ equations

✓ solve for parameters

From ① & ③, $\lambda = 4,\ \mu = 3$ parameters different ∴ do not meet.

✓ solve for a

$\therefore -5+8 = a-3$

$a = 6$

✓ identify P

$$P = \begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix}$$

or $P = 12i + 3j - 5k$.

(b) Show that the point $(1, -5, 4)$ lies in the plane containing the two lines. (3 marks)

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix}$$

✓ normal

$$\begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -5 \\ -1 \end{pmatrix} = -28 \qquad \text{Plane } r \cdot \begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix} = -28$$

✓ dot product for k.

$$\begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -28$$

✓ confirms equation is true for given point

$\therefore \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$ lies on the plane.

Question 16 **(7 marks)**

(a) Point A has coordinates $(8, -3, 3)$ and plane Π has equation $2x - y + 2z = 16$. Determine

(i) a vector equation for the straight line through A perpendicular to Π. (1 mark)

$$\underset{\sim}{l} = \begin{pmatrix} 8 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

(ii) the perpendicular distance of A from Π. (3 marks)

$$\begin{pmatrix} 8+2\lambda \\ -3-\lambda \\ 3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 16$$

✓ intersect line & plane

$$16 + 4\lambda + 3 + \lambda + 6 + 4\lambda = 16$$

✓ solve for λ

$$9\lambda = -9$$

$$\lambda = -1$$

✓ apply magnitude of normal

$$\left|\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\right| = 3 \quad \therefore \text{distance is } 3$$

(or determine point on plane & distance).

(b) Prove that the perpendicular distance from the origin to the plane $\mathbf{r} \cdot \mathbf{n} = k$ is $\dfrac{k}{|\mathbf{n}|}$. (3 marks)

Line through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ perpendicular to plane is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \underset{\sim}{n}$, or just $\underset{\sim}{l} = \lambda \underset{\sim}{n}$

✓ intersect line and plane

Using the method from (aii) above,

$$\lambda \underset{\sim}{n} \cdot \underset{\sim}{n} = k$$

✓ use $\underset{\sim}{n} \cdot \underset{\sim}{n} = |\underset{\sim}{n}|^2$ to rewrite λ

$$\lambda = \frac{k}{\underset{\sim}{n} \cdot \underset{\sim}{n}} = \frac{k}{|\underset{\sim}{n}|^2}$$

✓ apply $|\underset{\sim}{n}|$ & simplify

$$\text{Distance} = \lambda |\underset{\sim}{n}| = \frac{k}{|\underset{\sim}{n}|^2} |\underset{\sim}{n}| = \frac{k}{|\underset{\sim}{n}|}$$

Question 17 **(8 marks)**

Sphere S has diameter PQ, where P and Q have coordinates $(2, -3, 1)$ and $(-4, 7, 5)$ respectively.

(a) Determine the vector equation of the sphere. (3 marks)

Centre $C = \dfrac{P+Q}{2} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

✓ centre

radius $= |\overrightarrow{PC}|$

$= \left| \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right|$

✓ radius

✓ equation

$= \sqrt{38}$

$S: \left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right| = \sqrt{38}$

(b) Show that the point $(1, -1, 2)$ lies inside the sphere. (2 marks)

$\left| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right| = \sqrt{14}.$

✓ distance to point from centre

$\sqrt{14} < \sqrt{38}$ ∴ inside the sphere.

✓ compares.

(c) Show that the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is tangential to the sphere. (3 marks)

$\left| \begin{pmatrix} 1+\lambda \\ -3+\lambda \\ -3+\lambda \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right| = \sqrt{38}.$

✓ intersect

$(2+\lambda)^2 + (-5+\lambda)^2 + (-6+\lambda)^2 = 38$

✓ solve for λ

$3\lambda^2 - 18\lambda + 27 = 0$

✓ interpret.

$(\lambda - 3)^2 = 0$

$\lambda = 3$

A unique intersection point means that the line is tangential to the sphere.

Question 18 **(8 marks)**

Let $f(x) = \sqrt{x-1}$, $g(x) = \dfrac{3}{x}$ and $h(x) = f \circ g(x)$.

(a) Determine an expression for $h(x)$ and show that the domain of $h(x)$ is $0 < x \leq 3$. (3 marks)

$$h(x) = \sqrt{\frac{3}{x} - 1}$$

✓ expression for $h(x)$

$\frac{3}{x} - 1 \geq 0$ and $x \neq 0$

✓ show $x > 0$

$\frac{3}{x} \geq 1$ (note that this can never be true for $x < 0$)

✓ show $x \leq 3$

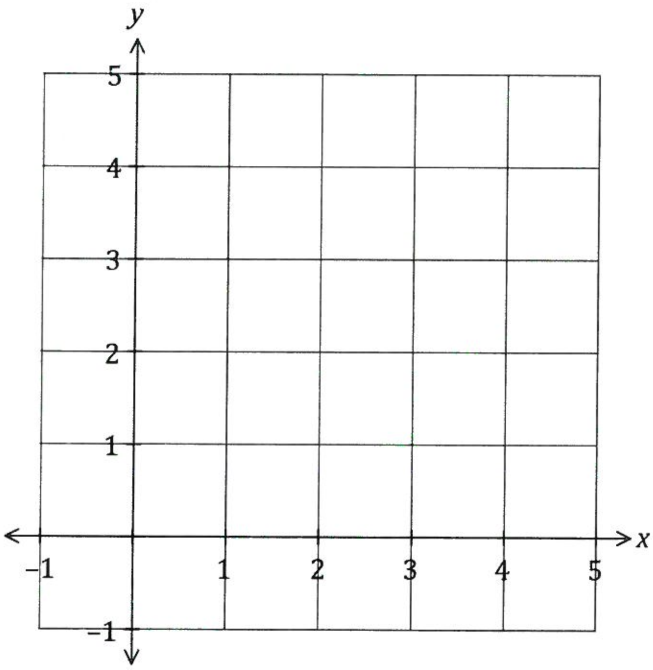
$3 \geq x$

$\therefore$ Domain of $h(x)$ is $0 < x \leq 3$

(b) Determine an expression for $h^{-1}(x)$, the inverse of $h(x)$. (1 mark)

$$h^{-1}(x) = \frac{3}{x^2+1}$$

(c) Sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the axes below. (3 marks)



(d) Solve $h(x) = h^{-1}(x)$, correct to 0.01 where necessary. (1 mark)

$x \simeq 0.38$, $x \simeq 1.21$, $x \simeq 2.62$

Question 19 **(8 marks)**

If $f(x) = \frac{1}{x+2}$ and $g(x) = x^2 - 5$,

(a) Determine the domain and range of the composition $f(g(x))$. (4 marks)

Domain $g(x)$ $x \in \mathbb{R}$. $x \neq \pm\sqrt{3}$

Range $g(x)$ $y \geq -5$ $y \neq -2$

$\Rightarrow$ Domain $f(g(x))$ $x \geq -5$ $x \neq -2$

Range $f(g(x))$ $y \leq \frac{-1}{3}$

$y > 0$

✓ link R g to D f.

✓ link not D f to R g.

✓ domain

✓ range

So for $f(g(x))$.

Domain $x \neq \pm\sqrt{3}$

Range $y > 0 \cup y \leq \frac{-1}{3}$

(b) The restriction $x \leq 0$ is applied to the composite function $f(g(x))$ so that its inverse exists. Determine the equation of this inverse and state the domain and range. (4 marks)

Inverse $y = -\sqrt{\frac{1}{x} + 3}$

✓ inverse.

✓ correct choice of $\pm\sqrt{\ }$

Domain such that $\frac{1}{x} + 3 \geq 0$, $x \neq 0$

$x > 0$, $x \leq -\frac{1}{3}$

✓ Domain

Range $y \geq 0$, $y \neq \sqrt{3}$

✓ Range